Scale invariance and invariant scaling in a mixed hierarchical system

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We consider a mixed hierarchical model with heterogeneous and monotone conditions of destruction. We investigate how scaling properties of defects in the model are related with heterogeneity of rules of destruction, determined by concentration of the mixture. The system demonstrates different kinds of criticality as a general form of system behavior. The following forms of critical behavior are obtained: stability, catastrophe, scale invariance, and invariant scaling. Different slopes of the magnitude-frequency relation are realized in areas of critical stability and catastrophe. A simple relation between the slope of magnitude-frequency relation and parameters of the mixture is established. [S1063-651X(99)08610-9]

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I. INTRODUCTION

Self-organized criticality, introduced by Bak, Tang, and Wiesenfeld, reflects a stable form of the self-similarity observed in the behavior of various multiscale systems [1]. In seismology the power-law form of the magnitude-frequency relationship is well known as the Gutenberg-Richter law [2,3], which characterizes scaling properties of seismicity both for the world seismicity and for different seismic regions:

$$\log_{10}N(M)=a-bM.$$

Here $N(M)$ denotes the number of earthquakes with the magnitude M. The slope of the magnitude-frequency relation b is close to unity for the world seismicity [3], but it takes different values in various seismic regions [4]. Recently, the Gutenberg-Richter law is usually associated with the self-organized criticality of the seismic process [5].

For any abstract or natural system the evolution of which is characterized by events of different scales, the power-law form of the magnitude-frequency relation means the self-similarity of the distribution of events:

$$p(l+1)=\lambda p(l),$$

where $p(l)$ denotes the density of events at scale l, and factor λ is determined by the slope of the magnitude-frequency relation. A special case of the unity slope $b=1$ defines the scale invariance of events:

$$p(l)=\text{const}$$

for all scales of the system.

In the avalanche sandpile models [1] the self-organized criticality is characterized by a linear magnitude-frequency relation with the slope equal to unity. When the self-organized criticality appears as a result of a stable fixed point of the transition function [6,7] it means the scale invariance of the system and the slope of the magnitude-frequency relation in this case is close to unity. Deviations from the unity slope exist only for a few low scale levels, when the system is far enough from the attractive critical state. A similar result was obtained in [8]. When the self-organized criticality appears as a result of a feedback relation which attracts the trajectory to a critical point, the slope of the magnitude-frequency relation depends on parameters of healing [9]. Various slopes of the magnitude-frequency relation were obtained in [10] because of complex interactions between different kinds of movements. However, the model is very complicated and it is difficult to see the origin of different scalings in this system.

To clarify the origin and conditions of different kinds of scaling we consider a simple hierarchical model of destruction, the has no temporal evolution and which behavior of which is governed by a small number of parameters. A similar model was suggested in [11], the unstable scale invariance was established in the single point of phase transition from stability to catastrophic behavior. The stable scale invariance (self-organized criticality) was observed in a similar model with nonmonotone [6] or heterogeneous [7] conditions of destruction. Although simple static hierarchical models are very abstract, they often demonstrate similar features as a complicated dynamical systems (compare, for example, [11] with [9] and [6] with [8]) but allow a clearer and simpler description. In the present investigation we consider a mixed hierarchical model suggested in [7]. It was shown in [7] that stable scale invariance (self-organized criticality) in this model is a result of strong heterogeneity of destruction conditions. Now we shall demonstrate that the self-organized criticality with different scaling λ is a general form of behavior of such systems.

The description of the model is given in Sec. II; elementary kinds of system behavior are described in Sec. III. In Secs. IV–VI we show the relation between the observed scaling and the involved heterogeneity for elementary kinds of system behavior; more complicated and more general cases are considered in Secs. VII and VIII. Results and possible application are discussed in Sec. IX.

II. GENERAL DESCRIPTION OF THE MODEL

We consider a hierarchical system of elements with branching number n (Fig. 1). Each element at level $l+1$ is relevant to a group of n elements of the previous level l.

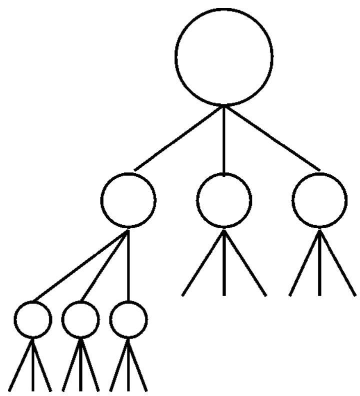

FIG. 1. Hierarchical system with branching number $n=3$.

Each element of the system may be in one of two possible states: broken or unbroken. An element in the broken state is referred to as a defect.

The state of an element at level $l+1$ is determined by number k of defects in the relevant group of n elements of the inferior level l. The rule which defines how the state of an element at level $l+1$ depends on the number k of defects on the relevant group of the previous level l is referred to as a condition of destruction. As usual, we assume that conditions of destruction are independent on level l, which means self-similarity of the model structure. However, this restriction does not determine self-similarity in the distribution of defects, but allows different kinds of system behavior, depending on concrete conditions of destruction [6–11,13]. The system behavior is described by concentrations of defects $p(l)$ at level l $(l=1,\ldots,L)$. Below we investigate statistical properties of densities of defects $p(l)$, when level l grows for different conditions of destruction.

Heterogeneity. When the condition of destruction is the same for all elements of the system, the model is called *homogeneous*. A *heterogeneous* model generally has different conditions of destruction for different elements. Thus, the homogeneous model is a degenerated case of a heterogeneous one.

Monotonicity. The model has *monotone* conditions of destruction, when each number of defects $k>k_0$ in a group of level l corresponds to a defect of superior level $l+1$, if k_0 defects in this group are relevant to the defect. When it is not true, the condition of destruction is *nonmonotone*.

It follows from the definition that homogeneous and monotone conditions of destruction may be defined by a lowest number k_0 of defects in a group of n elements which is relevant to a defect of a superior level. Consequently, there exist n (n is a branching number) different monotone and heterogeneous rules of destruction. In the case of nonmonotone and homogeneous conditions it is necessary to define a set of numbers k_i of defects relevant to the superior defect. This was the case for [6], where conditions of destruction were homogeneous and nonmonotone.

The importance of heterogeneity in relation with critical phenomena was previously noted [14]. We have considered in [7] a particular case of a monotone and heterogeneous system in and have shown that the scale invariance is related with high heterogeneity of the system. Now we investigate a more general case of such a model and describe other kinds of critical behavior obtained for heterogeneous and monotone conditions of destruction.

A. Critical numbers and concentrations of the mixture

The number k of defects in a group of n elements sufficient to bring forth a defect at the superior level is referred to as a critical number. Thus, an element of level $l+1$ is a defect if the relevant group of level l contains the critical number k or more defects. Any configuration with k or more defects in a group of n elements is referred to as a critical configuration.

In the homogeneous hierarchical model considered in [6] the critical number is the same for all elements of the system. The heterogeneous model suggested in [7] may be performed as a mixture of elements whose destruction is governed by different critical numbers k $(k=1,\ldots,n)$. In [7] the simplest case of $n=3$ was considered; here we investigate this model for arbitrary n.

The fraction of elements determined by the critical number k is denoted as a_k; the sum of concentrations a_k is equal to unity $(a_1+\cdots+a_n=1)$. It is assumed that fractions a_k do not change with level l. Concentrations a_k determine the heterogeneity of the system. A homogeneous system with critical number k corresponds to the degenerated case of the mixture, when only one concentration a_k is equal to unity and all other concentrations are zero.

B. Densities of defects

We denote as $p(l)$ the density of defects at level l. The density of defects at the first level $p(1)$ is a parameter of the model and densities of defects at higher levels may be calculated, when the density $p(1)$ and concentrations of the mixture a_k are fixed. The density of defects at level $l+1$ can be expressed from the density of defects at the previous level l as follows:

$$p(l+1)=F[p(l)], \tag{1}$$

where $F(p)$ denotes the probability to obtain critical configuration of defects in a group of n elements, if the probability of a defect is equal to p. We have assumed that concentrations a_k do not change with level, therefore the transition function F is the same for all levels of the system.

The density of configurations, containing exactly k defects in a group of n elements at level l, is equal to

$$W_k=C_n^k p^k(1-p)^{n-k}, \tag{2}$$

where $p=p(l)$ is the density of defects at level l, C_n^k denote the binomial coefficients, and n is a branching number. In a homogeneous system with critical number k the density of all critical configurations is equal to the sum of all configurations with k or more defects:

$$\Phi_k=\sum_{j=k}^{n} W_k .$$

In the heterogeneous case the transition function F is a weighted sum of Ψ_k, where each term of the sum is taken with the relevant concentration of the mixture a_k:

$$F(p)=\sum_{k=1}^{n} a_k\Phi_k .$$

Thus, the transition function F is completely determined by concentrations a_k:

$$F(p)=\sum_{k=1}^{n} C_n^k p^k(1-p)^{n-k}\sum_{i=1}^{k} a_i. \quad (3)$$

C. Magnitude-frequency relation

In studies of seismicity the magnitude of an earthquake is actually used as a measure of the energy of the earthquake. A linear relation between the magnitude of the earthquake and the linear size of its source is established [12]:

$$\log_{10}S\approx M+\text{const.} \quad (4)$$

A linear relation between the logarithm of the number of earthquakes and its magnitude, known as a Gutenberg-Richter law, is established for the world seismicity [3] and for particular seismic regions [4]:

$$\log_{10}N=a-bM. \quad (5)$$

We assume that the linear size of elements in the system falls with level $S(l)=S_0n^l$ and its number similarly grows, $N_e(l)=Cn^{L-l}$ (C denotes number of elements at the highest level L of the system). In seismology, the magnitude of earthquake is related with the linear size of the source area: $M\sim\log_{10}S$. Following [6–8,10], we consider the magnitude as a characteristic of the size of a defect at level l:

$$M(l)=l\log_{10}n \quad (6)$$

(we use a decimal logarithm in respect to the geophysical tradition). Expressing the average number of defects at the level l,

$$N(l)=Cn^{L-l}p(l), \quad (7)$$

we obtain from Eqs. (7) and (6) the magnitude-frequency relation for our model, which is an analog of Eq. (5) for seismicity and reads

$$\log_{10}N(l)=-M(l)+\log_{10}p(l)+\text{const.} \quad (8)$$

It is obvious that the form of the magnitude-frequency relation is completely determined by densities of defects $p(l)$ and by its evolution with level l. If densities of defects $p(l)$ tend to a constant value $p_0>0$, when level l grows, then the magnitude-frequency relation (8) is approximately linear with a slope equal to unity. A power form convergence of densities $p(l)$ to zero determine nonunity slopes of the magnitude-frequency relation. Both cases satisfy the Gutenberg-Richter law (5) and perform general forms of model behavior, as will be shown below.

D. Events

Some defects of level l correspond to a defect of the superior level $l+1$. In real observations the defect entering in a defect of the superior level cannot be detected. Only the defect of the highest range may be observed as an event. Therefore we define events as defects of level l which *do not enter* in a defect of the superior level $l+1$.

In the general case of the system, the transition function F is defined by Eq. (3) and the density of events at level $l<L$ is expressed as follows:

$$P(l)=\frac{1}{n}\sum_{k=1}^{n} kC_n^k p^k(l)[1-p(l)]^{n-k}\left(1-\sum_{i=1}^{k} a_i\right). \quad (9)$$

At the highest level all defects are events:

$$P(L)=p(L). \quad (10)$$

Expressing the average number of events:

$$\nu(l)=Cn^{L-l}P(l) \quad (11)$$

the magnitude-frequency relationship takes the form, similar to Eq. (8),

$$\log_{10}\nu(l)=-M(l)+\log_{10}P(l)+\text{const.} \quad (12)$$

If densities of defects $p(l)$ tend to zero, when level l grows [see for example, Fig. 2(b)], then it may be easily obtained from Eq. (9) that densities of events also tend to zero [Fig. 2(c)] and have the same order as densities of defects:

$$P(l)\sim(1-a_1)p(l). \quad (13)$$

It follows from Eqs. (8), (12), and (13) that in this case the magnitude-frequency relationship has the same slope for both numbers of defects and numbers of events.

When densities of defects $p(l)$ tend to a constant value $0<p_0<1$, then densities of events $P(l)$ also tend to a constant value $0<P_0<1$, determined by taking $p=p_0$ in the right side of Eq. (9). Thus, the magnitude-frequency relation for events, Eq. (12), is linear with a slope equal to unity, exactly like the magnitude-frequency relation written for defects, Eq. (8).

It is really important to distinguish events from defects, when densities of defects $p(l)$ increase with level and tend to unity [Fig. 3(b)], because densities of events $P(l)$ in this case tend to zero [see Eq. (9) and 3(c)], for all levels l, excepting the highest one [Eq. (10) and Fig. 3(b)].

III. HETEROGENEITY AND CRITICALITY

In this section we consider all possible kinds of system behavior and its relation with heterogeneity of the system reflected in concentrations of the mixture a_k. We are especially interested in the behavior characterized by a linear form of the magnitude-frequency relation. Such system behavior is referred to as critical. We shall classify different kinds of critical behavior observed in this system.

The form of the magnitude-frequency relation (8) is determined by the behavior of densities of defects $p(l)$, when level l grows. The transition function F is monotone, Eq. (3), therefore there always exists a limit $0\leqslant\lim_{l\to\infty}p(l)\leqslant 1$, so there are only three possibilities: (1) $p(l)$ tend to zero, (2) $p(l)$ tend to a constant value $0<p_0<1$, and (3) $p(l)$ tend to unity.

The rate of convergence is actual for the form of the magnitude-frequency relation, when $p(l)$ tend to zero or unity [cases (1) and (3)]. When the limit p_0 is between 0 and

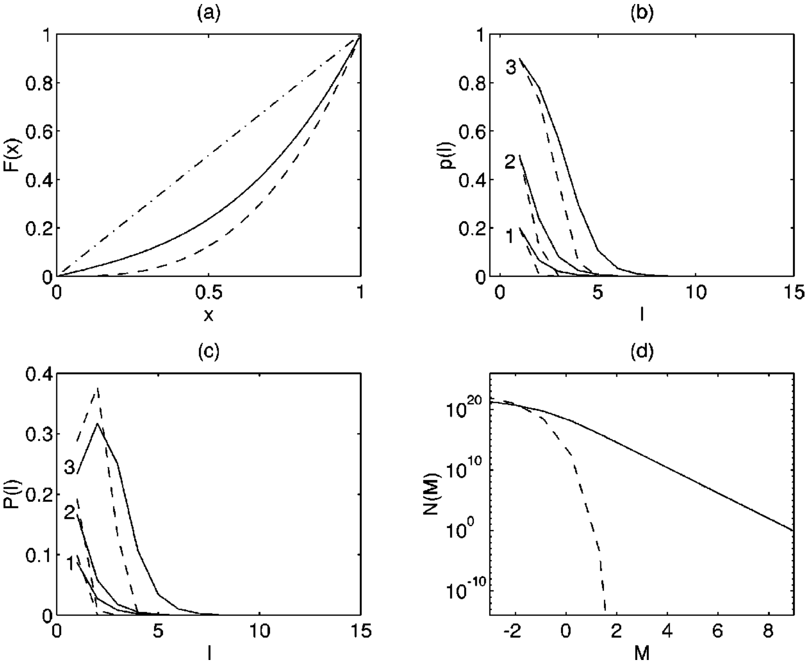

FIG. 2. Area of stability ($n=3$): (a) Transition function in the homogeneous case (dashed line) and general heterogeneous case (solid line) are below the dash-dotted diagonal line; (b) density of defects tends to zero for all values of $p(1)$: (1) 0.2, (2) 0.5, (3) 0.9; (c) density of events tends to zero for all p(1): (1) 0.2, (2) 0.2, (3) 0.8; (d) magnitude-frequency relation in homogeneous case (dashed line) has a downward bend, in the heterogeneous case (solid line) it is linear.

1 the magnitude-frequency relation is linear with a slope equal to unity and the rate of convergence has no influence on the value of the slope. We call case (1) *stability*, because there are no defects at high levels of the system; case (2) means the *scale invariance*; case (3) is called *catastrophe*, because the top level of the system is completely destroyed. Below we consider these three cases and all possible kinds of convergence for different parameters of the system and describe scaling properties of the system in each case.

IV. STABILITY

Let us begin with a specific degenerated homogeneous case, when the branching number $n=3$ and the critical number $k=n=3$ for all elements of the system. The transition function F is then the following:

$$F(p)=p^3. \tag{14}$$

As is shown in Fig. 2(a), the transition function lies below the diagonal line, therefore $F(p)<p$ for all values of p. The map F, defined by Eq. (14), has two fixed points: 0 and 1, the first one ($p=0$) is stable, the other one ($p=1$) unstable [Fig. 2(a), dashed line]. Thus, for all values of initial density of defects $p(1)$, densities of defects $p(l)$ and densities of events $P(l)$ tend to zero, when level l grows [Figs. 2(b) and 2(c) dashed lines]. The perturbation of the first level does not actually reach higher levels of the system, therefore this kind of behavior is referred to as a stability.

The relation between the density of defects $p(l)$ and the level l may be easily obtained from Eqs. (1) and (14):

$$p(l)=p(1)^{3^{l-1}}, \tag{15}$$

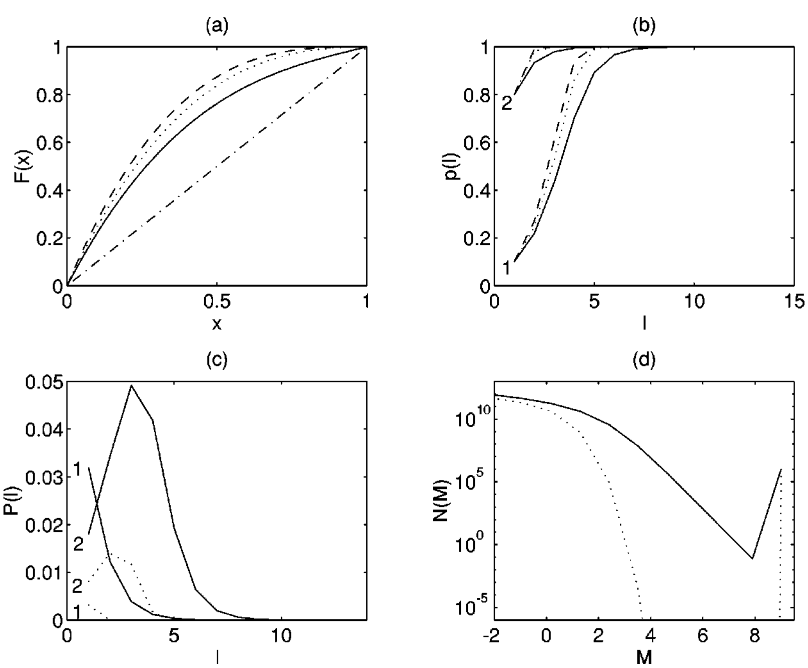

FIG. 3. Area of catastrophe ($n=3$): (a) Transition function is above the dash-dotted diagonal line in the homogeneous case (dashed line), degenerated heterogeneous case (dotted line), and general heterogeneous case (solid line); (b) density of defects tends to unity for all $p(1)$: (1) 0.1, (2) 0.9; (c) density of events tends to zero for all $p(1)$: (1) 0.1, (2) 0.9; (d) magnitude-frequency relation has a jump for highest magnitude, it is linear for $M>3$ in the general case (solid line) and has a downward bend in the degenerated heterogeneous case (dotted line).

where $p(1)$ denotes the density of defects at the first level of the system.

The logarithm of the total number of defects at level l is thus expressed from Eq. (7) as follows:

$$\log_{10}N(l) = -l\,\log_{10}n + \log_{10}p(l) + \text{const.} \tag{16}$$

So, we obtain from Eqs. (15) and (16) the exponential relation between the logarithm of the number of defects $N(l)$ and the scale level l:

$$\log_{10}N(l) = -l\,\log_{10}3 + 3^{l-1}\log_{10}p(1) + \text{const.} \tag{17}$$

It follows from Eqs. (8) and (17) that the magnitude-frequency relation has an exponential downward bend [Fig. 2(d), dashed line]:

$$\log_{10}N(M) = -M + \beta 3^{-\alpha M} + \text{const,} \tag{18}$$

where β and α are positive.

Critical stability

Now we consider a nondegenerate case of the mixture, close to the homogeneous case of stability. Let us consider the system with a branching number $n=3$, when elements with critical numbers $k=1$, $k=2$, and $k=3$ are mixed with concentrations a_1, a_2, and a_3, respectively. The concentration a_3 is close to unity, and concentrations a_1 and a_2 are close, but not exactly equal to zero.

Similarly to the previous case, the transition function F is posited below the diagonal line and has two fixed points: the stable point $p=0$ and the unstable one, $p=1$ [Fig. 2(a), solid line]. Thus, densities of defects $p(l)$ and events $P(l)$ tend to zero, when level l grows, for all values of initial density $p(1)$ [Figs. 2(b) and 2(c) solid lines]. From Eqs. (1) and (3) in the first order of $p(l)$ we obtain

$$\frac{p(l+1)}{p(l)} \sim 3a_1. \tag{19}$$

It follows from Eqs. (7), (6), (8), and (19) that the magnitude-frequency relation has a linear form with a slope $\log_{10}a_1/\log_{10}3$ [Fig. 2(d), solid line]:

$$\log_{10}N(M) = \frac{\log_{10}a_1}{\log_{10}3}M + \text{const.} \tag{20}$$

The linear form of the magnitude-frequency relation means the critical behavior of the system, therefore this kind of behavior is referred to as a *critical stability*. It follows from Eqs. (19) and (20) that in the area of stability the system demonstrate critical behavior, when $a_1 > 0$. Thus, the critical stability is a general case of system behavior and noncritical stability is a degenerate case.

In the simplest case of branching number $n=3$ the parametric area of critical stability may be easily described. Two conditions must be fulfilled by the transition function F, in order to obtain stability behavior: no fixed points exist inside the interval (0,1), excepting 0 and 1; the fixed point 0 is stable. It was shown in [7] that for $n=3$ concentrations of the mixture satisfy two conditions: $a_1 < 1/3$ and $(a_1 + a_2) < 2/3$ (Fig. 4).

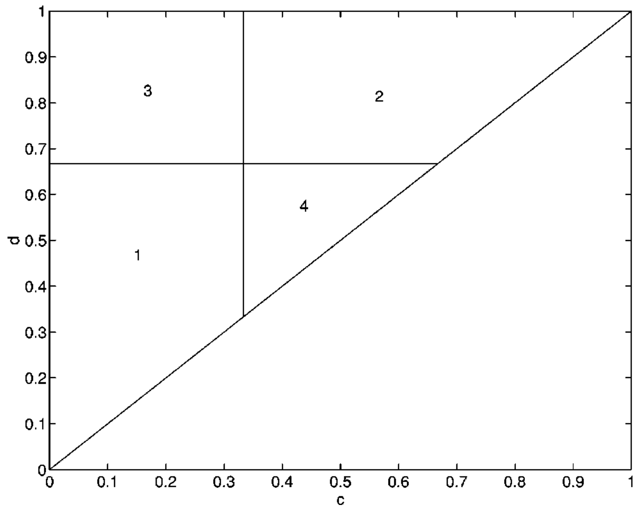

FIG. 4. Different parametric areas of system behavior for $n=3$. The mixture is parametrized by two free parameters: $c=a_1$ and $d=a_1+a_2$. There are four areas of system behavior: 1 stability; 2 catastrophe; 3 unstable scale invariance; 4 stable scale invariance.

V. CATASTROPHE

Let us always begin with a specific degenerated homogeneous case, when the branching number $n=3$ and the critical number $k=n=1$ for all elements of the system. The transition function F is then the following:

$$F(p) = 1-(1-p)^3. \tag{21}$$

As is shown in Fig. 3(a) (dashed line), the transition function is posited above the diagonal line, therefore $F(p) > p$ for all values of p. The map F, defined by Eq. (21), has two fixed points: 0 and 1, the first one ($p=0$) is unstable, the other one ($p=1$) is stable. Thus, for all values of initial density of defects $p(1)$, densities of defects $p(l)$ tend to unity, when level l grows [Fig. 3(b), dashed lines]. High levels of the system are almost destroyed, therefore this kind of behavior is referred to as a catastrophe. Densities of events are all equal to zero, $P(l)=0$ [see Eq. (9)], excepting the density of events of the highest level $P(L)=p(L)\approx 1$. Thus, the catastrophic behavior in the homogenous case is equivalent to a single event of the highest level.

A. Delocalization

Let us consider a heterogeneous degenerated case of the system with branching number $n=3$ when concentration of the mixture a_3 is equal to zero. Concentration a_1 is close to unity and concentration a_2 is not far from zero. It is also a case of catastrophic behavior: densities of defects $p(l)$ tend to unity [Fig. 3(b), dotted lines], when level l grows, and densities of events $P(l)$ tend to zero [Fig. 3(c), dotted lines]. The magnitude-frequency relation, considered for events, demonstrates a strong downward bend and a peak at the highest level [Fig. 3(d), dotted line]. Such behavior produces a gap in the interval of magnitudes before the highest ones that is similar to the delocalization phenomenon, observed in the Burridge-Knopoff model [15], in lattice models [16], and in a generalized dynamical hierarchical model [13], and associated with characteristic earthquakes [17].

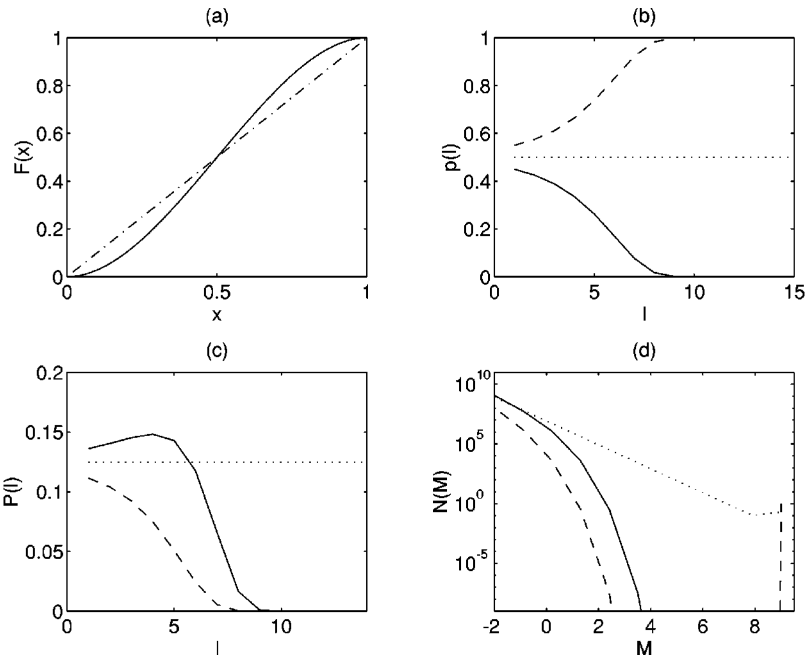

FIG. 5. Unstable scale invariance: (a) Transition function for $n=3$ ($a_1=0$, $a_2=1$); (b) density of defects; (c) density of events; (d) magnitude-frequency relation for events. Different lines correspond to different values of initial density of defects: $p(1)=0.45$, solid line; $p(1)=p_0=0.5$, dotted line; $p(1)=0.55$, dashed line.

B. Critical catastrophe

Let us now consider a nondegenerate case of the mixture close to the catastrophic one. In the system with branching number $n=3$, concentrations of the mixture are taken as follows: a_1 is close to unity, a_2 and a_3 are not far from zero.

The transition function F is posited above the diagonal line and has two fixed points: the unstable point $p=0$ and the stable one, $p=1$ [Fig. 3(a), solid line]. Thus, densities of defects $p(l)$ tend to unity and densities of events $P(l)$ tend to zero, when level l grows, for all values of initial density $p(1)$ [Fig. 3(b), solid lines]. From Eqs. (1), (3), and (9) in the first order of $1-p(l)$ we obtain

$$\frac{P(l+1)}{P(l)} \sim 3a_3. \tag{22}$$

It follows from Eqs. (11), (6), (12), and (22) that the magnitude-frequency relation has a linear form with a slope $\log_{10}a_3/\log_{10}3$ [Fig. 3(d), solid line]:

$$\log_{10}\nu(M) = \frac{\log_{10}a_3}{\log_{10}3}M + \text{const.} \tag{23}$$

The linear form of the magnitude-frequency relation means the critical behavior of the system, therefore this kind of behavior is referred to as a critical catastrophe. It follows from Eqs. (22) and (23), that in the area of catastrophe the system demonstrates critical behavior, when $a_3>0$. Thus, similarly to stability, the critical catastrophe is a general case of system behavior and noncritical catastrophe is a degenerate case.

It was obtained in [7] that for the simplest case of branching number $n=3$ the parametric area of critical catastrophe is determined by two conditions imposed on concentrations a_k of the mixture: $a_1>1/3$ and $(a_1+a_2)>2/3$ (Fig. 4).

VI. SCALE INVARIANCE

Let us suppose that transition function F has a fixed point p_0 inside the interval (0,1) and different from zero and unity: $0<p_0<1$. When the initial intensity of defects $p(1)$ is equal to the value of the fixed point $p(1)=p_0$, then all densities of defects take the same value $p(l)=p_0$ for all levels of the system. Thus, the system demonstrates the scale invariance. It follows from Eq. (8) that the magnitude-frequency relation for the case of scale invariance is linear with a slope equal to unity, which means the critical behavior of the system. It follows from Eqs. (9) and (12) that the linear form with a unity slope of the magnitude-frequency relation exists also when events instead of defects are considered. Thus, the scale invariance represents a specific form of critical behavior, characterized by a unity slope of the magnitude-frequency relation. In both cases of critical stability and catastrophe, considered above, the slope of the magnitude-frequency relation was greater than unity [see Eqs. (20) and (23)].

A. Unstable scale invariance

In the simplest system with branching number $n=3$ the transition function F defined by Eq. (3) always has two fixed points, $p=0$ and $p=1$. It has a third fixed point $0<p=p_0<1$, when one of two following pairs of conditions for concentrations of the mixture are fulfilled [7]:

$$a_1<1/3 \text{ and } a_1+a_2>1/3, \tag{24}$$

$$a_1>1/3 \text{ and } a_1+a_2<1/3. \tag{25}$$

In the first case (24) the fixed point $p=p_0$ is unstable; in the second case (25) it is stable. In this section we consider the case where the unstable fixed point $p=p_0$ exists.

It follows from Eqs. (3) and (24) that both fixed points $p=0$ and $p=1$ of the transition function F are stable [Fig. 5(a)]. Thus, $F(p)<p$ for all $p<p_0$ and densities of defects

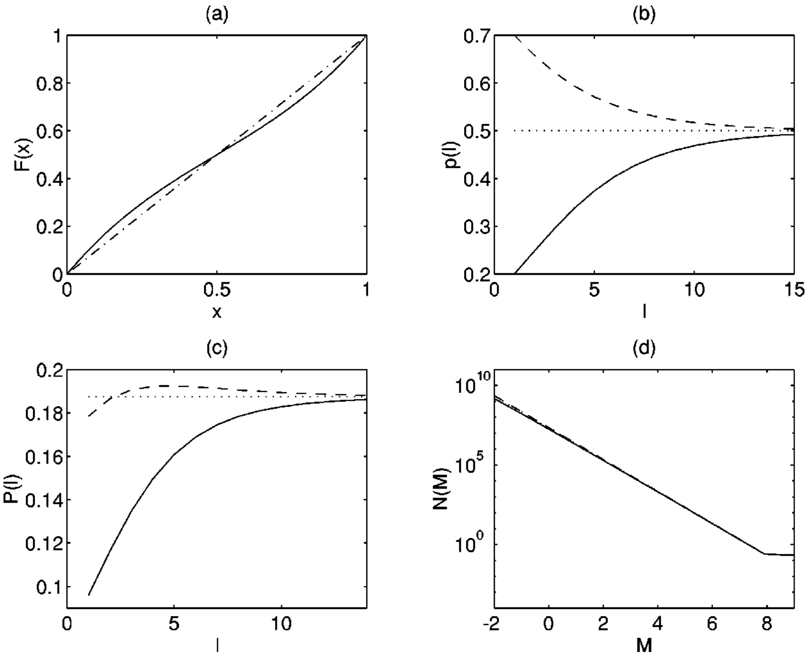

FIG. 6. Stable scale invariance: (a) Transition function for $n=3$ ($a_1=0.5$, $a_2=0$); (b) density of defects; (c) density of events; (d) magnitude-frequency relation for events. Different lines correspond to different values of initial density of defects: $p(1)=0.2$, solid line; $p(1)=p_0=0.5$, dotted line; $p(1)=0.7$, dashed line. For all $p(1)$ the magnitude-frequency relation is linear with a slope equal to unity.

$p(l)$ defined by Eq. (1) tend to zero, when level l grows, for all values of initial densities $p(1)<p_0$ [Fig. 5(b), solid line]. Densities of events also tend to zero for $p(1)<p_0$ [Fig. 5(c), solid line]. If $p>p_0$, then $F(p)>p$; thus, for all $p(1)>p_0$ densities of defects $p(l)$ tend to unity and densities of events $P(l)$ tend to zero, when level l grows [Figs. 5(b) and 5(c), dashed lines]. The point $p=p_0$ is fixed for the transition function F, therefore $p(l)=p_0$ and $P(l)=P_0$ for all levels l, if $p(1)=p_0$ [Figs. 5(b) and 5(c), dotted lines]. This kind of behavior is referred to as the unstable scale invariance (Fig. 4). As was shown, in the simplest case of $n=3$, the unstable scale invariance is related with the phase transition from stability [when $p(1)<p_0$] to catastrophic behavior (when $p(1)>p_0$). The scale invariance is obtained in the single point $p(1)=p_0$ and only this point is characterized by the unity slope of the magnitude-frequency relation [Fig. 5(d)].

B. Stable scale invariance

Let us consider the system with branching number $n=3$, when conditions (25) for concentration of the mixture are fulfilled. It follows from Eqs. (3) and (25) that the transition function F has two unstable fixed points, $p=0$ and $p=1$, and the stable one, $p=p_0$ [Fig. 6(a)]. Thus for all values of initial density of defects $p(1)$ densities of defects $p(l)$ defined by Eq. (1) tend to the value p_0, when level l grows [Fig. 6(b)]. Similarly, densities of events $P(l)$ tend to a constant value P_0 [Fig. 6(c)]. Thus, the scale invariance may be observed for all values of $p(1)$ and the magnitude-frequency relation has the asymptotically linear form with a slope equal to unity [Fig. 6(d)]. This kind of behavior is referred to as the stable scale invariance or the self-organized criticality (Fig. 4). In the area of stable scale invariance the slope of magnitude-frequency relation is equal to unity for all values of parameters a_k and $p(1)$.

Concentrations of the mixture a_k reflect heterogeneity of the considered system: in homogenous systems one concentration is equal to unity, others are zero; in the system with strong heterogeneity concentrations a_k are far from unity. It follows from conditions (25) that stable scale invariance cannot be achieved in the homogenous monotone model: concentrations of the mixture relevant to the most different critical numbers (a_1 and a_3 in the considered case) must be greater than 1/3. Thus, the stable scale invariance in the behavior of the system indicates high heterogeneity, numerically described by corresponding conditions for concentrations a_k of the mixture.

VII. FIXED POINTS AND THE SYSTEM BEHAVIOR: GENERAL CASE

Let us consider a general case of the transition function $F(p)$ defined by Eq. (3). The transition function $F(p)$ is a sum of monotone functions $\Phi(p)$ with positive coefficients a_k, therefore it also monotonically increases with p, therefore the function F monotonically increases, and the derivation $F'(p)$ is positive inside the interval (0,1).

Let us consider, the transition function $F(p)$ with m fixed points $p=p_i$, where $i=1,\ldots,m$. It follows from Eq. (3), that $p=0$ and $p=1$ are the fixed points of the map F, thus $p_1=0$ and $p_m=1$. It is known that a fixed point p_i is stable when the absolute value of the derivation $F'(p_i)$ is less than unity, and it is unstable, when $|F'(p_i)|$ is greater than unity. The derivation F' is positive, therefore p_i is stable, when $F'(p_i)<1$ and it is unstable when $F'(p_i)>1$.

Let us consider two consequent fixed points of the map F: p_i and p_{i+1}. Fixed points of the map $F(p)$ are zero points of the map $F(p)-p$. There is no zero of the map $F(p)-p$ between p_i and p_{i+1}, therefore the derivation $[F(p)-p]'=F'(p)-1$ has different signs in p_i and p_{i+1}. Thus, if $F'(p_i)<1$, then $F'(p_{i+1}>1$ and vice versa. Therefore stable and unstable fixed points of the map F alternate.

It follows from Eq. (1) and monotone increasing of the transition function F that densities of defects $p(l)$ monotonically tend to a limit value p^*, which, generally, depends on the initial density of defects $p(1)$. It follows from the continuity of the transition function F that this limit point $p=p_0$ must be a fixed point of the map $F(p)$. Thus, the number of possible kinds of behavior demonstrated by the system with transition function F is completely determined by the number of fixed points of the map F. When the fixed point $0<p_i<1$ is unstable, then it governs the unstable scale invariance for $p(1)=p_i$. If the fixed point $0<p_i<1$ is stable, then it determine the area of the stable scale invariance [the self-organized criticality (SOC)]. The area of stability exists if the fixed point $p_1=0$ is stable. The area of catastrophe exists if the fixed point $p_m=1$ is stable. The unstable scale invariance is realized in all unstable fixed points $p(1)=p_i$, different from 0 and 1. Thus, all kinds of system behavior (stability, catastrophe, stable, and unstable scale invariance) may be realized in the behavior of one system, if the corresponding transition function has sufficient number of fixed points. As illustrations, we suggest below two examples of this complex behavior for the system with branching number $n=5$.

A. Complex case: stability-SOC-catastrophe

The transition function for the system with $n=5$ is determined by five concentrations of the mixture a_k, whose sum is equal to unity. To reduce the number of free parameters we consider a symmetrical case, when $a_1=a_5$ and $a_2=a_4$. So, the transition function is completely determined by two concentrations of the mixture, for example, a_1 and a_2:

$$F(p)=5a_1p(1-p)^4+10(a_1+a_2)p^2(1-p)^3$$
$$+10[1-(a_1+a_2)]p^3(1-p)^2$$
$$+5(1-a_1)p^4(1-p)+p^5. \tag{26}$$

In the symmetric case the transition function always has three fixed points: 0, 0.5, and 1. It can be easily obtained that five fixed points of the map F exist, when

$$10(a_1+a_2)+15a_1-7>0, \text{ and } 5a-1<0 \text{ or}$$

$$10(a_1+a_2)+15a_1-7<0, \text{ and } 5a-1>0.$$

When $10(a_1+a_2)-5a_1-3>0$ and $5a_1-1<0$ (Fig. 7, area 4), the transition function F has five fixed points: $p=0$, $p=0.5$, and $p=1$ are stable; $p=p_0$ and $p=1-p_0$ are unstable [Fig. 8(a)]. The area of stability corresponds to initial densities of defects $0\leqslant p<p_0$, densities of defects $p(l)$ tend to zero, when level l grows [Fig. 8(b), curve 1]. The area of stable scale invariance (SOC) corresponds to $p_0<p(1)<1-p_0$, and densities of defects $p(l)$ tend to 0.5, when level l grows [Fig. 8(b), curves 2,3]. The area of catastrophe corresponds to $1-p_0<p(1)\leqslant 1$, densities of defects tend to 1 [Fig. 8(b), curve 4). The unstable scale invariance is realized for $p(1)=p_0$ and $p(1)=1-p_0$. Thus, when the initial density $p(1)$ increases, the system passes all possible kinds of behavior.

B. Scale invariance: phase transition SOC-SOC

A nontrivial case of scale invariant behavior may be obtained in the same model, when $10(a_1+a_2)+15a_1-7<0$ and $5a_1-1>0$ [Fig. 7, area 3]. The transition function has five fixed points [Fig. 8(c)]: two stable ($p=p_0$ and $p=1-p_0$) and three unstable ($p=0$, $p=0.5$, and $p=1$). The existence of two stable fixed points determines stable scale invariance (SOC) for all values of initial density $p(1)$, excepting the unstable fixed points 0, 0.5, and 1 [Fig. 8(d)]. Nevertheless, when $p(1)=0$, $p(1)=0.5$, or $p(1)=1$, the system also demonstrates the scale invariance, but the unstable one. Thus, in this case for all values of the initial density $p(1)$, the scale invariance may be observed. However, this is not trivial stable scale invariance, because densities of defects $p(l)$ tend to two stable limits, depending on initial density $p(1)$ [Fig. 8(d)].

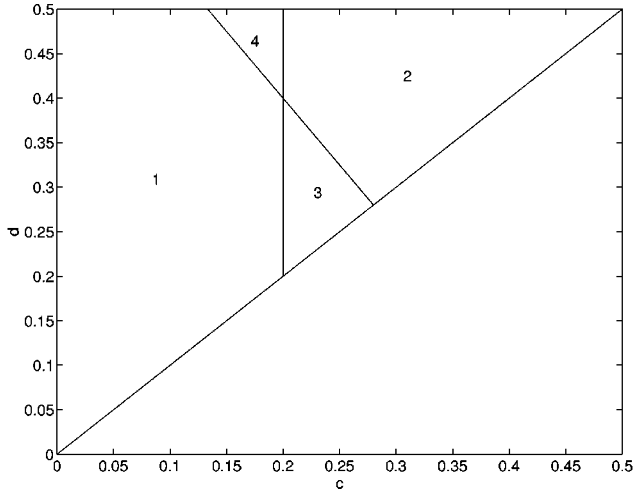

FIG. 7. Different parametric areas of system behavior for symmetric case with $n=5$. The mixture is parametrized by two free parameters $c=a_1=a_5$ and $d=a_1+a_2=a_4+a_5=1-a_3$. There are four areas: 1, unstable scale invariance (phase transition from stability to catastrophe); 2, stable scale invariance; 3, nontrivial scale invariance (phase transition from one stable point to another); 4, all possible kinds of primary system behavior (stability, unstable scale invariance, stable scale invariance, catastrophe) are realized by changing only initial density $p(1)$.

VIII. INVARIANT SCALING

Let us consider a specific case, when all concentrations of the mixture a_k are equal: $a_k=1/n$. After the substitution of a_k into Eq. (3), we obtain the transition function F:

$$F(p)=p. \tag{27}$$

Thus, it follows from Eq. (1) that densities of defects $p(l)$ are equal to one another for all levels l: $p(l)=p(1)$, which means the scale invariance exists for all values of initial density $p(1)$, no special scale may be distinguished. The magnitude-frequency relation is always linear with a slope equal to unity, but this is not appropriate to any special concentration of destruction at the highest level of the system. The system does not reach any special critical state, but it is critical in any state. This kind of behavior is referred to as the invariant scaling.

It seems that invariant scaling is a degenerated case of system behavior; only one point in the space of parameters a_k exactly corresponds to the invariant scaling. However, when parameters a_k are not very far from this point the value $F(p)$ is close to p for all p [Fig. 9(a)]. Therefore conver-

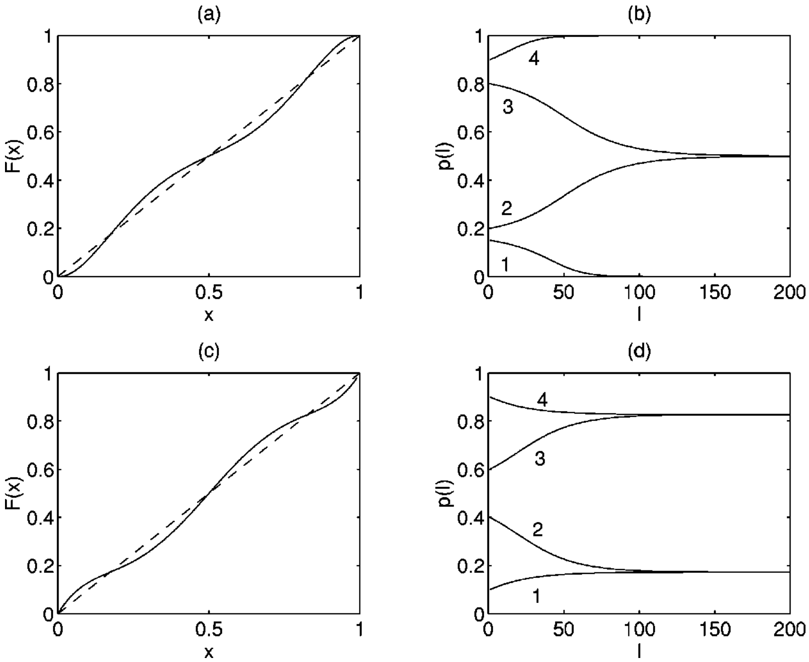

FIG. 8. Complex behavior for symmetric system with $n=5$. (a) Transition function ($a_1=a_5=0.18$, $a_2=a_4=0.28$) rescaled with factor 10 in order to see the difference from the diagonal (dashed) line. (b) Density of defects for the transition function (a) for different values of $p(1)$: (1) 0.15, (2) 0.2, (3) 0.8, (4) 0.9; (c) transition function ($a_1=a_5=0.22$, $a_2=a_4=0.12$) rescaled with factor 10; (d) density of defects for the transition function (c) for different values of $p(1)$: (1) 0.1, (2) 0.4, (3) 0.6, (4) 0.9.

gence of densities $p(l)$ to limits p_i is very slow and for a restricted number of levels it is hardly observed [Fig. 9(b)]. In fact, the deviation between concentrations of the mixture a_k and $1/n$ may be quite significant: $a_1=a_5=0.22, a_2=a_4=0.12$ for $n=5$ (Fig. 9). So, the density of defects $p(l)$ conserves its value $p(l)$ for several levels of the system that make this kind of behavior quite general.

IX. DISCUSSION AND CONCLUSIONS

We have suggested a model where the heterogeneity of destruction is governed by concentration of the mixture of different rules. This formalism may be applied both when the system contains elements of different strength, or when the stress field is heterogeneous and therefore different densities of fractures of smaller scales are necessary to build a fracture of the next range. Although the model is rather abstract and does not reflect particular features of any concrete system, it may be considered as a good illustration of the statistical properties of a wide class of multiscale systems, such as, for example, the fracturing of samples or earthquakes.

The simple hierarchical model considered above shows how scaling properties of the system may be related with the heterogeneity of conditions of destruction. In the homogeneous case this system demonstrates critical behavior only in the unstable critical point of phase transition [11]; it is necessary to involve nonmonotone rules of destruction in order to obtain stable critical behavior [6]. So, for the homogeneous monotone model, criticality is a degenerated behavior, and general behavior is noncritical.

The behavior of the system became critical when a heterogeneity of destruction is assumed. It was shown that the magnitude frequency is always linear in a log/log plot, when the relevant parameter of the mixture is nonzero. Thus, for the heterogeneous system the critical behavior is a general case and noncritical behavior is a degenerated case. For the lithosphere of the Earth, for example, it is rather natural to assume heterogeneous rules of destruction instead of homogeneous ones, therefore the power-law form of the Gutenberg-Richter law for earthquakes is more natural than the exponential one. This is the simplest explanation of the Gutenberg-Richter law, although it is very abstract.

We may distinguish three kinds of criticality in the behavior of the system: general self-similarity, which is characterized by a linear magnitude-frequency relation with various slopes; scale invariance, which is associated with equal probabilities of defects at all ranges of the system and unity slope of the magnitude-frequency relation; and invariant scaling, when the system does not reach any special critical state, but all states of the system are critical. These three cases are also related with different order of heterogeneity involved in the system. The general self-similarity exists for any nonzero

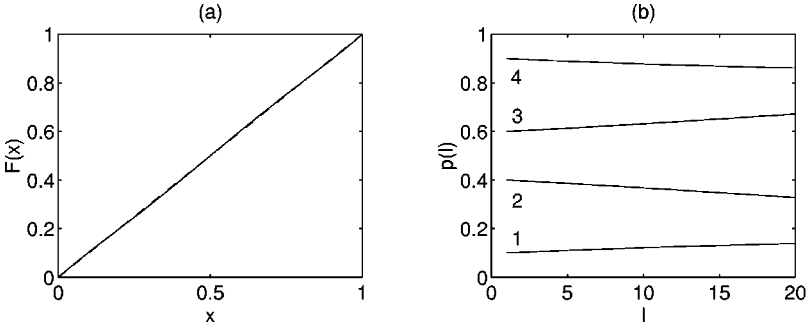

FIG. 9. Invariant scaling for a restricted number of levels. (a) Transition function for $n=5$, $a_1=a_5=0.22$, $a_2=a_4=0.12$ (not rescaled) is close to the diagonal (dashed) line. (b) Densities of defects for first 20 levels slightly change their initial value: 1, $p(1)=0.1$; 2, $p(1)=0.4$; 3, $p(1)=0.6$; 4, $p(1)=0.9$.

concentrations of the mixture. Concentrations of the mixture, which govern the heterogeneity, are reflected in the slope of the magnitude-frequency relation. In order to obtain stable scale invariance it is necessary to mix the most fragile and the most rigid elements in high proportions, which means high heterogeneity of the system. Invariant scaling emerges in the special situation when all concentrations of the mixture are equal, all rules of destruction are mixed in equal proportions. This is the most heterogeneous case of the mixture. Thus, the type of criticality reflects the heterogeneity of the system.

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